4. Limits and Continuity

In this lecture, we will discuss

- Limits
  - Review of limit for $f : \mathbb{R} \to \mathbb{R}$.
  - Limit of Functions of Several Variables
  - Limit of a Vector-Valued Function
  - Limit Laws
- Continuity
  - Review of continuity for $f : \mathbb{R} \to \mathbb{R}$.
  - Continuity of Functions of Several Variables
  - Properties of Continuous Functions

📌**General remark before we start.**

- The formal definition of the limit of functions uses the (ϵ, δ) language. We will state that for strictness throughout our notes, but we will not bother with proofs using this language in our course. You will revisit it later and learn more about the proofs in your first real-analysis class.
- The examples in the notes are related to some of your WebWork homework. The correspondence are indicated.

Limits

We first review what is a limit of a real-valued single variable function and its properties. Then, we generalize those notions to real-valued and vector-valued functions.

Review of limit for $f : \mathbb{R} \to \mathbb{R}$ (Real-valued single variable)

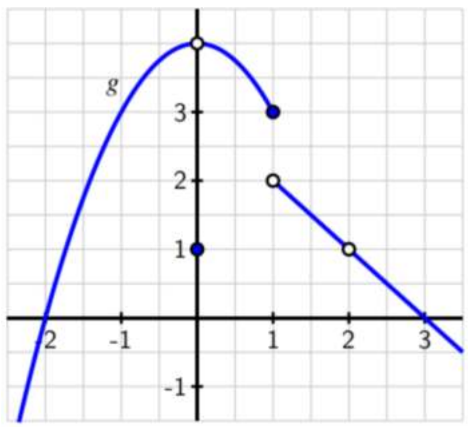
Definition Limit of a Function of One Variable

A function $f : \mathbb{R} \to \mathbb{R}$ has limit L as x approaches a, in symbols $\lim_{x\to a} f(x) = L$, if and only if for any given number $\epsilon > 0$ there is a number $\delta > 0$ such that

$$0 < |x-a| < \delta \quad \text{implies} \quad |f(x) - L| < \epsilon.$$

One-side limit

Example 0. Let $g(x)$ is the function given by the graph below.



What are the values of $\lim\limits_{x\to 1^-} g(x)$, $\lim\limits_{x\to 1^+} g(x)$, $\lim\limits_{x\to 1} g(x)$, $\lim\limits_{x\to 0} g(x)$, and $\lim\limits_{x\to 2} g(x)$?

- As $x \to 1$ from left (but not equals to 1), we have $g(x) \to 3$.

  Thus $\lim\limits_{x\to 1^-} g(x) = 3$

- As $x \to 1$ from right (but not equals to 1), we have $g(x) \to 2$

  Thus $\lim\limits_{x\to 1^+} g(x) = 2$

- As $\lim\limits_{x\to 1^-} g(x) \neq \lim\limits_{x\to 1^+} g(x)$, $\lim\limits_{x\to 1} g(x)$ does not exist (DNE)

- For $\lim\limits_{x\to 0} g(x)$, we have $\lim\limits_{x\to 0} g(x) = \lim\limits_{x\to 0^-} g(x) \overset{?}{=} \lim\limits_{x\to 0^+} g(x) = 4$

- Similarly, we have $\lim\limits_{x\to 2} g(x) = 1$.

Limit Laws

Assume $f(x)$ and $g(x)$ are functions on $\mathbb{R}$ and $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist. Let x be any constant. Then

- $\lim_{x\to a}(f(x)+g(x)) = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$
- $\lim_{x\to a}(f(x)-g(x)) = \lim_{x\to a} f(x) - \lim_{x\to a} g(x)$
- $\lim_{x\to a} cf(x) = c\cdot \lim_{x\to a} f(x)$
- $\lim_{x\to a}(f(x)\cdot g(x)) = \lim_{x\to a} f(x)\cdot \lim_{x\to a} g(x)$
- $\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$ if $\lim_{x\to a} g(x) \neq 0$

Next, we will generalize the limit to the real-valued and vector-valued functions.

Limit of Functions of Several Variables

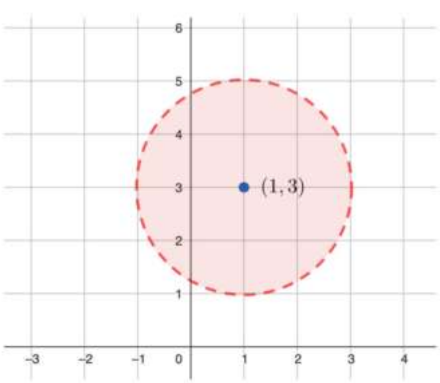
Definition. Open Balls in $\mathbb{R}^m$
The open ball $B(\mathbf{a}, r) \subseteq \mathbb{R}^m$ with center $\mathbf{a} = (a_1, \ldots, a_m)$ and radius $r (r > 0)$ is the set of all points $\mathbf{x}$ in $\mathbb{R}^m$ whose distance from a fixed point a is smaller than r. In symbols,

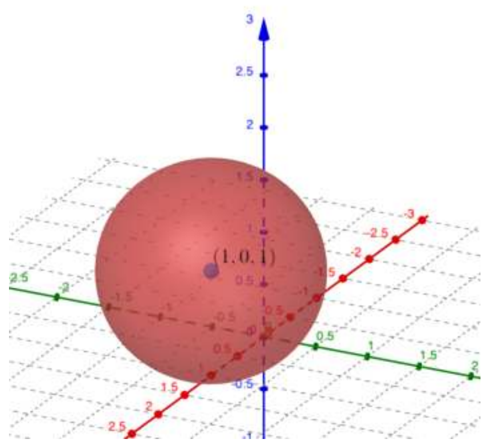
$$B(\mathbf{a}, r) = \{\mathbf{x} \in \mathbb{R}^m \mid \|\mathbf{x}-\mathbf{a}\| < r\},$$

where $\mathbf{x} = (x_1, \ldots, x_m)$ and $\|\mathbf{x}-\mathbf{a}\| = \sqrt{(x_1-a_1)^2 + \cdots + (x_m-a_m)^2}$.

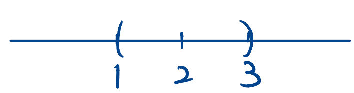
For example,

- In $\mathbb{R}^2$, the open ball $B((1,3),2)$ contains all points in $\mathbb{R}^2$ whose distance from $(1,3)$ is strictly smaller than 2.
- In $\mathbb{R}^3$, the open ball $B((1,0,1),1)$ contains all points in $\mathbb{R}^3$ whose distance from $(1,0,1)$ is strictly smaller than 1.





- In $\mathbb{R}$, the open ball $B((2),1)$ contains all points in $\mathbb{R}$ whose distance from 2 is strictly smaller than 1. Note it is simply the open interval $(1,3)$.



Definition. Limit of a Real-Valued Function of Several Variables

Let $f : U \subseteq \mathbb{R}^m \to \mathbb{R}$ be a real-valued function of m variables. We say that the limit of $f(\mathbf{x}) = f(x_1, \ldots, x_m)$ as $\mathbf{x} = (x_1, \ldots, x_m)$ approaches $\mathbf{a} = (a_1, \ldots, a_m)$ is L, in symbols

$$\lim_{\mathbf{x}\to\mathbf{a}} f(\mathbf{x}) = L \qquad \text{or} \qquad \lim_{(x_1,\ldots,x_m)\to(a_1,\ldots,a_m)} f(x_1, \ldots, x_m) = L,$$

if and only if for every $\epsilon > 0$ there is a number $\delta > 0$ such that $\mathbf{x} \in U$ and

$$0 < \|\mathbf{x} - \mathbf{a}\| < \delta \quad \text{implies} \quad |f(\mathbf{x}) - L| < \epsilon.$$

Remark. Notice that when $m = 1$, we recover precisely the definition of the Limit of a Function of One Variable we recalled earlier in the notes.

Example 1 (Related to WebWork HW #9 and 10)

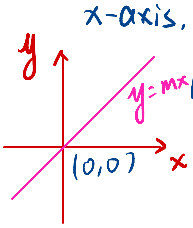
Find the limit, if it exists, or explain why if it does not exist.

$$\lim_{(x,y)\to(0,0)} \frac{2xy}{2x^2 + y^2} \qquad f : \mathbb{R}^2 \to \mathbb{R}$$

ANS: Note we can not simply substitute $(x,y) = (0,0)$ into the function, since it gives us $\frac{0}{0}$.

To show the limit does not exist, we use the idea discussed on the left.

- First, suppose $(x,y) \to (0,0)$ along the x-axis, thus $y=0$ along the path, we have

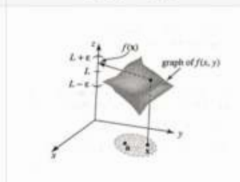
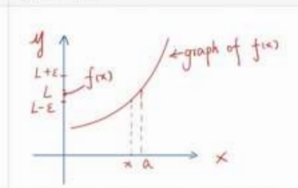
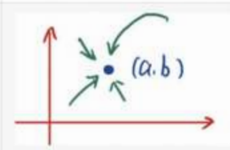
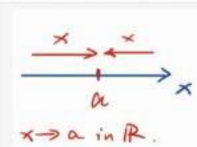


$$\lim_{(x,y)\to(0,0)} \frac{2xy}{2x^2+y^2} = \lim_{x\to 0} \frac{0}{2x^2} = 0$$

- Then, we take a line $y = mx$ through the origin.

$$\lim_{(x,y)\to(0,0)} \frac{2xy}{2x^2+y^2} = \lim_{x\to 0} \frac{2m\cancel{x^2}}{2\cancel{x^2}+m^2\cancel{x^2}} = \frac{2m}{2+m^2}$$

Since the value depends on any $m \in \mathbb{R}$ it follows that different approaches give different result, the limit does not exist.

	$f:\mathbb{R}^2 \to \mathbb{R}$	$f:\mathbb{R} \to \mathbb{R}$
Limit of f		
Ways to approach a point		

- Note in $\mathbb{R}^2$, there are infinitely many ways to approach to a point $\vec{a} = (a,b)$
- The limit is independent of the way we approach to the point $\vec{a} = (a,b)$
- If the limit exists, it is a unique number.
- Thus if two different ways give us two different candidates for the limit, we can conclude the limit does not exist.

Limit of a Vector-Valued Function

Definition Limit of a Vector-Valued Function $\vec{F}: \mathbb{R}^m \to \mathbb{R}^n$

Let $\mathbf{F}(\mathbf{x}) = (F_1(\mathbf{x}), \ldots, F_n(\mathbf{x}))$ be a vector-valued function of m variables, and let $\mathbf{a} = (a_1, \ldots, a_m)$ and $\mathbf{L} = (L_1, \ldots, L_n)$. We say that the function $\mathbf{F}(\mathbf{x})$ has limit $\mathbf{L}$ as $\mathbf{x}$ approaches $\mathbf{a}$, and write $\lim_{\mathbf{x}\to\mathbf{a}} \mathbf{F}(\mathbf{x}) = \mathbf{L}$, if and only if

$$\lim_{\mathbf{x}\to\mathbf{a}} F_1(\mathbf{x}) = L_1, \ldots, \lim_{\mathbf{x}\to\mathbf{a}} F_n(\mathbf{x}) = L_n.$$

In other words, the limit of a vector-valued function is computed componentwise:

$$\lim_{\mathbf{x}\to\mathbf{a}} \mathbf{F}(\mathbf{x}) = \left(\lim_{\mathbf{x}\to\mathbf{a}} F_1(\mathbf{x}), \ldots, \lim_{\mathbf{x}\to\mathbf{a}} F_n(\mathbf{x})\right),$$

provided that all limits on the right side (and those are limits of real-valued functions) exist.

Example 2 (Related to WebWork HW #3)

Find the limit of the vector-valued function $\mathbf{F}(t) = \left(\frac{e^{2t}-1}{t}, \frac{t^3}{t^4-t^3}, \frac{\sin 3t}{t}\right)$ at $t = 0$.

ANS: By the previous def, we know

$$\lim_{t\to 0} \vec{F}(t) = \lim_{t\to 0}\left(\frac{e^{2t}-1}{t}, \frac{t^3}{t^4-t^3}, \frac{\sin 3t}{t}\right)$$

$$= \left(\lim_{t\to 0}\frac{e^{2t}-1}{t}, \lim_{t\to 0}\frac{t^3}{t^4-t^3}, \lim_{t\to 0}\frac{\sin 3t}{t}\right)$$

Use L'Hospital Rule, we know

type $\frac{0}{0}$

L'Hôpital's rule

L'Hôpital's rule states that for functions f and g which are differentiable on an open interval I except possibly at a point c contained in I, if $\lim_{x\to c} f(x) = \lim_{x\to c} g(x) = 0$ or $\pm\infty$, and $g'(x) \neq 0$ for all x in I with $x \neq c$, and $\lim_{x\to c}\frac{f'(x)}{g'(x)}$ $\lim_{x\to c}\frac{f'(x)}{g'(x)}$ exists, then $\lim_{x\to c}\frac{f(x)}{g(x)} = \lim_{x\to c}\frac{f'(x)}{g'(x)}$

$$\lim_{t\to 0}\frac{e^{2t}-1}{t} = \lim_{t\to 0}\frac{(e^{2t}-1)'}{t'} = \lim_{t\to 0}\frac{2e^{2t}}{1} = 2$$

$$\lim_{t\to 0}\frac{t^3}{t^4-t^3} = \lim_{t\to 0}\frac{\cancel{t^3}\cdot 1}{\cancel{t^3}(t-1)} = \lim_{t\to 0}\frac{1}{t-1} = -1$$

$$\lim_{t\to 0} \frac{\sin 3t}{t} \overset{\text{L'Hospital}}{=\!=} \lim_{t\to 0} \frac{3\cos 3t}{1} = 3$$

type $\frac{0}{0}$

Thus $\lim_{t\to 0} \vec{F}(t) = (2, -1, 3)$

Limit Laws

The computation of a limit can be simplified by using the limit laws.

Theorem 1. Limit Laws

Let $\mathbf{F}, \mathbf{G} : \mathbb{R}^m \to \mathbb{R}^n$, $f, g : \mathbb{R}^m \to \mathbb{R}$ and assume that $\lim\limits_{\mathbf{x}\to\mathbf{a}} \mathbf{F}(\mathbf{x})$, $\lim\limits_{\mathbf{x}\to\mathbf{a}} \mathbf{G}(\mathbf{x})$, $\lim\limits_{\mathbf{x}\to\mathbf{a}} f(\mathbf{x})$ and $\lim_{\mathbf{x}\to\mathrm{a}} g(\mathbf{x})$ exist. Then

(a) $\lim\limits_{\mathbf{x}\to\mathbf{a}}(\mathbf{F}(\mathbf{x}) + \mathbf{G}(\mathbf{x}))$ and $\lim\limits_{\mathbf{x}\to\mathbf{a}}(\mathbf{F}(\mathbf{x}) - \mathbf{G}(\mathbf{x}))$ exist and

$$\lim_{\mathbf{x}\to\mathbf{a}}(\mathbf{F}(\mathbf{x}) \pm \mathbf{G}(\mathbf{x})) = \lim_{\mathbf{x}\to\mathbf{a}} \mathbf{F}(\mathbf{x}) \pm \lim_{\mathbf{x}\to\mathbf{a}} \mathbf{G}(\mathbf{x}).$$

(b) $\lim\limits_{\mathbf{x}\to\mathrm{a}} f(\mathbf{x})g(\mathbf{x})$ and $\lim_{\mathbf{x}\to\mathbf{a}} c\mathbf{F}(\mathbf{x})$ (for any constant c) exist, and

$$\lim_{\mathbf{x}\to\mathbf{a}}(f(\mathbf{x})g(\mathbf{x})) = \left(\lim_{\mathbf{x}\to\mathbf{a}} f(\mathbf{x})\right)\left(\lim_{\mathbf{x}\to\mathbf{a}} g(\mathbf{x})\right) \quad \text{and} \quad \lim_{\mathbf{x}\to\mathbf{a}}(c\mathbf{F}(\mathbf{x})) = c \lim_{\mathbf{x}\to\mathbf{a}} \mathbf{F}(\mathbf{x}).$$

(c) If $\lim_{\mathbf{x}\to\mathbf{a}} g(\mathbf{x}) \neq 0$, then $\lim\limits_{\mathbf{x}\to\mathbf{a}} \dfrac{f(\mathbf{x})}{g(\mathbf{x})}$ exists, and

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim\limits_{x\to a} f(x)}{\lim\limits_{x\to a} g(x)}.$$

(d) For any $\mathbf{a} \in \mathbb{R}^m$ and any constant $\mathbf{c} \in \mathbb{R}^n$,

$$\lim_{\mathbf{x}\to\mathbf{a}} \mathbf{x} = \mathbf{a} \text{ and } \quad \lim_{\mathbf{x}\to\mathbf{a}} \mathbf{c} = \mathbf{c}.$$

In part (d) the symbol $\mathbf{c}$ denotes the function $\mathbf{F} : \mathbb{R}^m \to \mathbb{R}^n$ given by $\mathbf{F}(\mathbf{x}) = \mathbf{c}$ for all $\mathbf{x} \in \mathbb{R}^m$.

Example 3 (Related to WebWork HW #6)

Find the limit.

(1) $\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{\sqrt{(x^2-y^2+1)}-1}$

ANS: We first rationalize the denominator

$$\lim_{(x,y)\to(0,0)} \frac{(x^2-y^2)(\sqrt{x^2-y^2+1}+1)}{(\sqrt{x^2-y^2+1}-1)(\sqrt{x^2-y^2+1}+1)}$$

$$= \lim_{(x,y)\to(0,0)} \frac{(\cancel{x^2-y^2})(\sqrt{x^2-y^2+1}+1)}{\cancel{x^2-y^2}\cancel{+1}\cancel{-1}}$$

$$= \lim_{(x,y)\to(0,0)} \sqrt{x^2-y^2+1}+1 = \sqrt{0-0+1}+1 = 2$$

(2) $\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^2-y^2} = 0$

ANS: Using the polar coordinates $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$

As $(x,y) \longrightarrow (0,0)$, $r \to 0$. we have

$$\lim_{r\to 0} \frac{(r\cos\theta)^2 r\sin\theta}{r^2(\cos^2\theta-\sin^2\theta)}$$

$$= \lim_{r\to 0} r \cdot \frac{\cos^2\theta\sin\theta}{\cos^2\theta-\sin^2\theta} = 0$$ no matter

what θ is.

Continuity

Review of limits for $f : \mathbb{R} \to \mathbb{R}$.

Definitions of continuity

Recall that a function $f : \mathbb{R} \to \mathbb{R}$ is *continuous* at $x = a$ if and only if

1. $\lim\limits_{x \to a} f(x)$ exists,
2. f is defined at a, and
3. $\lim\limits_{x \to a} f(x) = f(a)$.

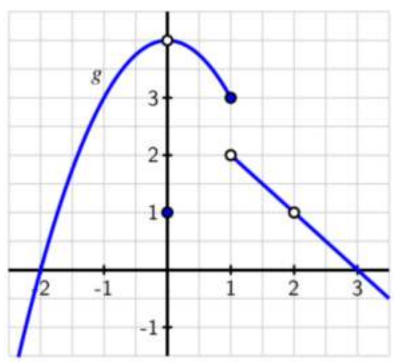
- A function f is continuous on an interval (c, d) if it is continuous at every point a in (c, d).
- A function f is continuous on a closed interval $[c, d]$ if it is continuous on (c, d) and $\lim\limits_{x \to c^+} f(x) = f(c)$ and $\lim\limits_{x \to d^-} f(x) = f(d)$.

Recall **Example** 0, where $g(x)$ as the following graph.

1. At $x=0$, $\lim\limits_{x \to 0} g(x) \neq g(0)$

2. At $x=1$, $\lim\limits_{x \to 1} g(x)$ DNE

3. At $x=2$, $g(2)$ is not defined



By definition, we know $g(x)$ is not continuous at points $0, 1, 2$.

Properties of Continuous Functions

If $f(x)$ and $g(x)$ are continuous at any real value c over the closed interval $[a, b]$, then the following are also continuous at any real value c over the closed interval $[a, b]$:

- $f(x) + g(x)$
- $f(x) - g(x)$
- $f(x)g(x)$
- $\dfrac{f(x)}{g(x)}$, as long as $g(c) \neq 0$.

The composition of continous functions is also continuous:

Let $f : A \to \mathbb{R}$ and $g : B \to \mathbb{R}$ where $f(A) \subset B$. If f is continuous at $c \in A$ and g is continuous at $f(c) \in B$, then $g \circ f : A \to \mathbb{R}$ is continuous at c.

Examples of Continuous Functions

- Polynomial Functions ($x^3 + x^2 + 3$, $2x^2 + 4x$, etc)
- Trigonometric Functions in certain periodic intervals $(\sin \mathrm{x}, \cos \mathrm{x}, \tan \mathrm{x}$, etc.)
- Exponential Functions (e^{2x}, e^{x+3x})
- Logarithmic Functions in their domains ($\ln(1+3x)$, $\log_3 6x$, etc.)

Next, we will generalize the definition of continuous functions to the real-valued and vector-valued functions.

Continuity of Functions of Several Variables

Definition. Continuity of Functions of Several Variables

A function $f : U \subseteq \mathbb{R}^m \to \mathbb{R}$ is continuous at $\mathbf{x} = \mathbf{a}$ if and only if

1. $\lim\limits_{\mathbf{x} \to \mathbf{a}} f(\mathbf{x})$ exists,
2. f is defined at $\mathbf{a}$, and
3. $\lim\limits_{\mathbf{x} \to \mathbf{a}} f(\mathbf{x}) = f(\mathbf{a})$.

We say that f is continuous on a set U (or just f is continuous) if and only if it is continuous at all points in U.

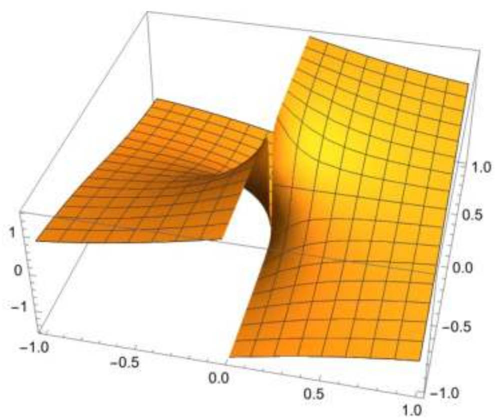
A vector-valued function $\mathbf{F} = (F_1, \ldots, F_n) : U \subseteq \mathbb{R}^m \to \mathbb{R}^n$ is continuous if and only if its components $F_i, i = 1, \ldots, n$, are continuous.

Remark. When $n = m = 1$, we recover our definition for $f(x)$ to be continuous.

An Example of a Function Not Continuous at $x = 0$.

The following graph of $f(x, y) = \arctan(y/x)$ is not continuous at points where $x = 0$. If you have Mathematica installed, you can type the following code to get a 3D plot.

```
Plot3D[{ArcTan[y/x]}, {x, -1, 1}, {y, -1, 1}]
```



Example 4 (Related to WebWork HW #9)

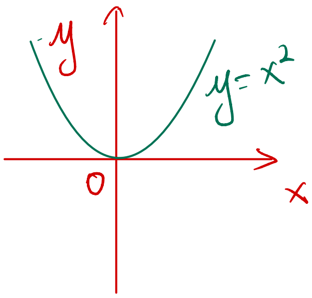
Consider

$$f(x,y)=\begin{cases}\dfrac{x^2y}{x^4+y^2} & \text{if } (x,y)\neq(0,0)\\ 0 & \text{if } (x,y)=(0,0)\end{cases}$$

(1) Show that $\displaystyle\lim_{(x,y)\to(0,0)}\frac{x^2y}{x^4+y^2}$ does not exist.

(2) Is $f(x,y)$ continuous at $(0,0)$?

ANS: (1) We use the idea discussed in Example 1.



- First, we choose the approach along y-axis. (that is $x=0$), we have

$$\lim_{(x,y)\to(0,0)}\frac{x^2y}{x^4+y^2}=\lim_{y\to0}\frac{0}{y^2}=0$$

- Then we choose the approach along the line $y=mx$, then

$$\lim_{(x,y)\to(0,0)}\frac{x^2y}{x^4+y^2}=\lim_{x\to0}\frac{mx^3}{x^4+m^2x^2}=\lim_{x\to0}\frac{mx}{x^2+m^2}=0$$

But there still are other ways to approach to $(0,0)$!

If we approach the point (x,y) along the parabola $y=x^2$, we have

$$\lim_{(x,y)\to(0,0)}\frac{x^2y}{x^4+y^2}=\lim_{x\to0}\frac{x^4}{x^4+x^4}=\lim_{x\to0}\frac{x^4}{2x^4}=\frac{1}{2}$$

Thus we get different values of $\dfrac{x^2y}{x^4+y^2}$ if we choose different ways of approaching to the point $(0,0)$.

So the limit $\lim_{(x,y)\to(0,0)} \frac{x^2 y}{x^4+y^2}$ DNE.

(2) Since the $\lim_{(x,y)\to(0,0)} f(x,y)$ DNE

$f(x,y)$ does not satisfy #1 in the definition of continuity

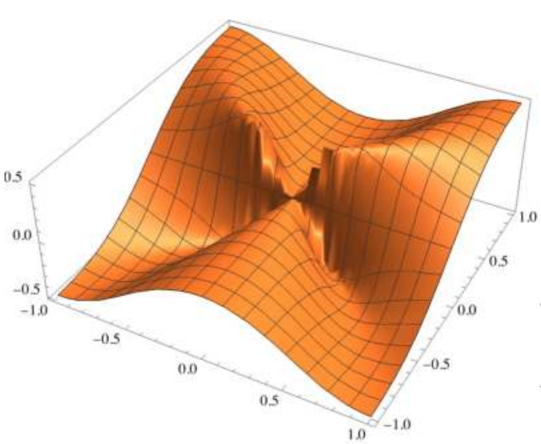
So $f(x,y)$ is not continuous at $(0,0)$

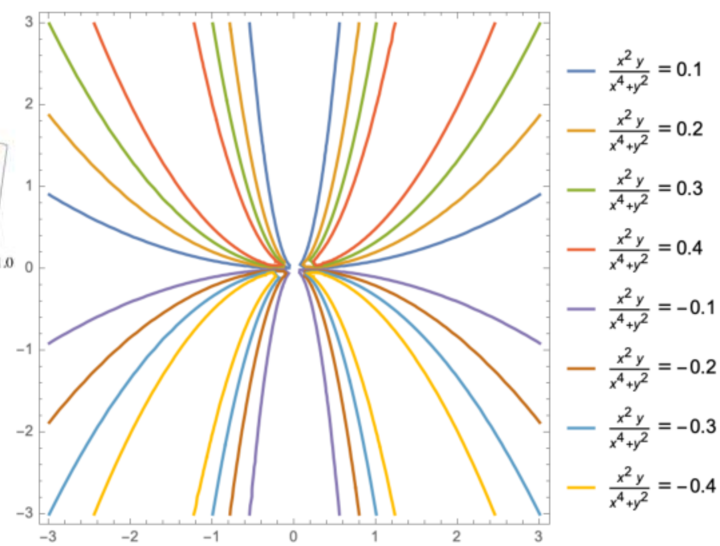
Digression on using computer software to plot the function $f(x, y)$.

We can also use the computer to check the behavior of the function $f(x, y) = \dfrac{x^2y}{x^4 + y^2}$. For example, we can use the following Mathematica code to get the graph of $f(x, y)$ and its contour curves with some pre-chosen values of $z = \dfrac{x^2y}{x^4 + y^2}$:

```
(*Graph of f(x,y)*)
Plot3D[{(x^2*y)/(x^4 + y^2)}, {x, -1, 1}, {y, -1, 1},
 PlotTheme -> "Scientific"]
(*Contour Curves of f(x,y) for certain vlues of z=f(x,y)*)
ContourPlot[{x^2*y/(x^4 + y^2) == 0.1, x^2*y/(x^4 + y^2) == 0.2,
  x^2*y/(x^4 + y^2) == 0.3, x^2*y/(x^4 + y^2) == 0.4,
  x^2*y/(x^4 + y^2) == -0.1, x^2*y/(x^4 + y^2) == -0.2,
  x^2*y/(x^4 + y^2) == -0.3, x^2*y/(x^4 + y^2) == -0.4}, {x, -3,
  3}, {y, -3, 3}, PlotLegends -> "Expressions"]
```

We get the following graphs after running the code. Note that it is hard to tell from the first graph what happens to $f(x, y)$ near the point $(x, y) = (0, 0)$. The second contour graph suggests the limit of $f(x, y)$ does not exist at the point $(0, 0)$.





Properties of Continuous Functions

The continuous functions behave as we expect:

Theorem 2. Properties of Continuous Functions

Let $\mathbf{F}, \mathbf{G} : U \subseteq \mathbb{R}^m \to \mathbb{R}^n (n \geq 1)$ and $f, g : U \subseteq \mathbb{R}^m \to \mathbb{R}$ be continuous at a $\in U$. Then

(a) the functions $\mathbf{F} \pm \mathbf{G}$, defined by $(\mathbf{F} \pm \mathbf{G})(\mathbf{x}) = \mathbf{F}(\mathbf{x}) \pm \mathbf{G}(\mathbf{x})$, are continuous at a.

(b) the function $c\mathbf{F}$, defined by $(c\mathbf{F})(\mathbf{x}) = c\mathbf{F}(\mathbf{x})$, is continuous at $\mathbf{a}$.

(c) the function fg, defined by $(fg)(\mathbf{x}) = f(\mathbf{x})g(\mathbf{x})$, is continuous at $\mathbf{a}$.

(d) the function f/g, defined by $(f/g)(\mathbf{x}) = f(\mathbf{x})/g(\mathbf{x})$, is continuous at a, if $g(\mathbf{a}) \neq 0$.

Theorem 3. Continuity of Composition of Functions

Let $\mathbf{F} : U \subseteq \mathbb{R}^m \to \mathbb{R}^m$ and $\mathbf{G} : V \subseteq \mathbb{R}^n \to \mathbb{R}^p$ be such that the range $\mathbf{F}(U)$ of $\mathbf{F}$ is contained in the domain V of $\mathbf{G}$, so that the composition $\mathbf{G} \circ \mathbf{F}$ is defined. If $\mathbf{F}$ is continuous at $\mathbf{a}$ and $\mathbf{G}$ is continuous at $\mathbf{b} = \mathbf{F}(\mathbf{a})$, then $\mathbf{G} \circ \mathbf{F}$ is continuous at $\mathbf{a}$.

Exercise 5 (Related to Webwork HW #7)

Find the largest set on which the function $f(x, y) = \sqrt{x + 3y} + \sqrt{x - 3y}$ is continuous.

ANS: We need to find the set such that both $\sqrt{x+3y}$ and $\sqrt{x-3y}$ are continuous

That is $\begin{cases} x+3y \geq 0 \\ x-3y \geq 0 \end{cases} \Rightarrow \begin{cases} y \geq -\frac{x}{3} \\ y \leq \frac{x}{3} \end{cases}$

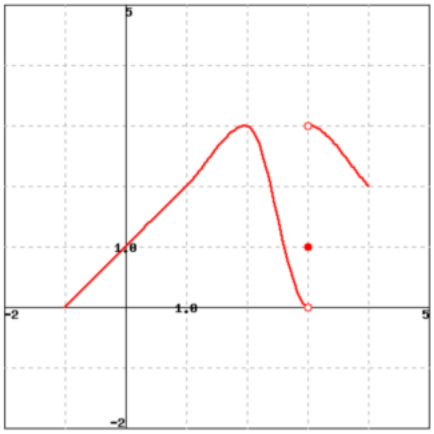
Thus we have the largest set

$$\left\{ (x,y) \,\middle|\, -\frac{x}{3} \leq y \leq \frac{x}{3} \right\}$$

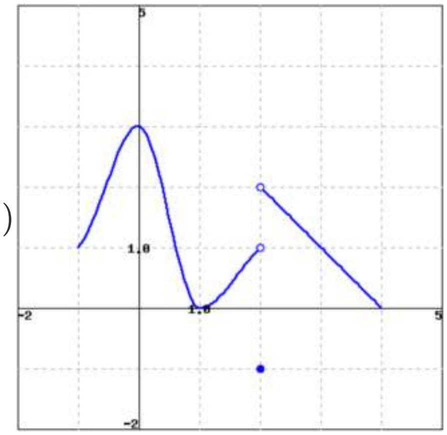
Exercise 6 (Related to Webwork HW #1)

The following question is a review of the limit of functions $\mathbb{R} \to \mathbb{R}$.

The graphs of $f(x)$ and $g(x)$ are given below. Use them to evaluate each quantity listed. Write DNE if the limit or value does not exist (or if it's infinity).



$f(x)$



$g(x)$

1. $\lim_{x\to 3^-}[f(x)g(x)] =$ 0
2. $\lim_{x\to 3^+}[f(x)g(x)] =$ 3
3. $f(g(3)) =$ 2
4. $\lim_{x\to 2^-}[f(x)g(x)] =$ 3
5. $f(3)/g(3) =$ 1
6. $\lim_{x\to 2^+}[f(x)/g(x)] =$ 1.5
7. $\lim_{x\to 2^+}[f(x)+g(x)] =$ 5
8. $f(3)+g(3) =$ 2
9. $\lim_{x\to 3^+}[f(x)+g(x)] =$ 4
10. $\lim_{x\to 2^+}[f(g(x))] =$ 3
11. $f(g(2)) =$ 0
12. $f(2)g(2) =$ -3
13. $\lim_{x\to 2^+}[f(x)g(x)] =$ 6
14. $\lim_{x\to 3^-}[f(x)/g(x)] =$ 0
15. $\lim_{x\to 3^-}[f(x)+g(x)] =$ 1
16. $\lim_{x\to 3^+}[f(g(x))] =$ 2
17. $\lim_{x\to 3^+}[f(x)/g(x)] =$ 3
18. $\lim_{x\to 3^-}[f(g(x))] =$ 2
19. $\lim_{x\to 2^-}[f(g(x))] =$ 2
20. $f(3)g(3) =$ 1
21. $f(2)/g(2) =$ -3

22. $\lim_{x \to 2^-} [f(x) + g(x)] =$ 4

23. $f(2) + g(2) =$ 2

24. $\lim_{x \to 2^-} [f(x)/g(x)] =$ 3